

# Inflationary universe from perfect fluid and $F(R)$ gravity and its comparison with observational data

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We investigate the descriptions for the observables of inflationary models, in particular, the spectral index of curvature perturbations, the tensor-to-scalar ratio, and the running of the spectral index, in the framework of perfect fluid models and  $F(R)$  gravity theories through the reconstruction methods. Furthermore, the perfect fluid and  $F(R)$  gravity descriptions of inflation are compared with the recent cosmological observations such as the Planck satellite and BICEP2 experiment. It is demonstrated with explicit examples that perfect fluid may lead to the inflationary universe consistent with the Planck data. It is also shown that several  $F(R)$  gravity models, especially, a power-law model gives the best fit values compatible with the spectral index and tensor-to-scalar ratio within the allowed ranges suggested by the Planck and BICEP2 results.

PACS numbers: 04.50.Kd, 98.80.-k, 98.80.Cq, 12.60.-i

## I. INTRODUCTION

Owing to the recent data taken by the BICEP2 experiment [1] on the tensor-to-scalar ratio of the primordial density perturbations, additionally to the observations by the satellites of the Wilkinson Microwave anisotropy probe (WMAP) [2, 3] and the Planck [4, 5], inflation has attracted much more attention.

The potential form of inflaton is related to the spectrum of the density perturbations generated during inflation [6]. Several models of inflation have recently been constructed to account for the Planck and BICEP2 data [7]. Particularly, scalar field models of inflation have been explored in comparison with the data analysis of the BICEP2 [8, 9].

Recently, in Ref. [10], we have explicitly performed the reconstruction of scalar field theories with inflation leading to the theoretical consequences compatible with the observational data obtained from the Planck and BICEP2 in terms of the spectral index of the curvature fluctuations, the tensor-to-scalar ratio, and the running of the spectral index. As the developments of these investigations, in this paper, we reconstruct the descriptions of inflation in perfect fluid models and  $F(R)$  gravity theories<sup>1</sup>. Especially, we re-express the observables of inflationary models, i.e., the spectral index of curvature perturbations, the tensor-to-scalar ratio, and the running of the spectral index, in terms of the quantities in perfect fluid models and  $F(R)$  gravity theories. We also compare the perfect fluid and  $F(R)$  gravity descriptions of inflation with the recent observational data obtained from the Planck satellite and BICEP2 experiment. An inflationary model of the fluid with its inhomogeneous viscosity has recently been discussed in Ref. [14]. Such reconstructions also work to realize bounce universes [15, 16]. The reconstruction of  $F(R)$  gravity model from observational data has been executed in Ref. [17]. We note that the first successful inflationary model in such a sort of modified gravity (i.e.,  $R^2$  inflation) has been proposed in Ref. [18]. There have also been existed its supergravity extension [19] and the related models [20]. We remark that there have been the attempts to make modified gravity models to explain the Planck and BICEP2 results [21–23]. Moreover, the features of the spectral

<sup>1</sup> For recent reviews on dark energy problem and modified gravity theories including  $F(R)$  gravity to solve it, see, for instance, Refs. [11–13].

index in induced gravity [24] and scalar-tensor theories [25] have been examined in detail.

We here emphasize the physical motivations of this work and explain the importance of formulations of the observables for inflationary models in terms of the perfect fluid and  $F(R)$  gravity. In ordinary scalar field model of inflation, the the spectral index, the tensor-to-scalar ratio, and the running of the spectral index are represented by using the potential of the scalar field. Therefore, we can judge what scalar field models are consistent with the observations. Similarly, if we re-formulate these observables for inflationary models in the perfect fluid and  $F(R)$  gravity descriptions, by comparing these theoretical representations with the observations, we can obtain information on the properties of the perfect fluid and  $F(R)$  gravity models to account for the observations in the early universe. Thus, the outcome of this approach is that since the natures of the perfect fluid and  $F(R)$  gravity models have been examined by using the observations about the current cosmic acceleration, by combining the insights obtained from our approach in terms of inflation, we can find novel conditions for the perfect fluid and  $F(R)$  gravity models to be viable from the cosmological point of view. We use units of  $k_B = c = \hbar = 1$  and express the gravitational constant  $8\pi G_N$  by  $\kappa^2 \equiv 8\pi/M_{\text{Pl}}^2$  with the Planck mass of  $M_{\text{Pl}} = G_N^{-1/2} = 1.2 \times 10^{19}$  GeV.

The organization of the paper is the following. In Sec. II, we write the slow-roll parameters and express the observables of inflationary models in the formulation of the perfect fluid. Next, in Sec. III we represent the slow-roll parameters and the observables of inflationary models with the quantities in  $F(R)$  gravity. These descriptions are also compared with the Planck and BICEP2 data. Conclusions are presented in Sec. IV. The explicit expressions of the observables for inflationary models in the description of the perfect fluid and those in the  $F(R)$  gravity description are given in Appendixes A and B, respectively. In Appendix C, we present the representations of the observables for inflationary models in the linear form of square of the Hubble parameter and those in the exponential one. In Appendix D, the relation between the equation of state (EoS) parameter and the tensor-to-scalar ratio is described.

## II. PERFECT FLUID DESCRIPTION OF THE SLOW-ROLL PARAMETERS

Usually, the slow-roll parameters are related to the potential of the scalar field, namely, inflaton. In this section, we explore how these parameters may be expressed in terms of perfect fluid.

### A. Slow-roll parameters

For the model of a scalar field  $\phi$  coupled with gravity

$$S = \int d^4x \sqrt{-g} \left( \frac{R}{2\kappa^2} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right), \quad (\text{II.1})$$

where  $R$  is the scalar curvature, we define the slow-roll parameters  $\epsilon$ ,  $\eta$  and  $\xi$  as follows,

$$\epsilon \equiv \frac{1}{2\kappa^2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2, \quad \eta \equiv \frac{1}{\kappa^2} \frac{V''(\phi)}{V(\phi)}, \quad \xi^2 \equiv \frac{1}{\kappa^4} \frac{V'(\phi)V'''(\phi)}{(V(\phi))^2}. \quad (\text{II.2})$$

Here and in the following, the prime means the derivative with respect to the argument such as  $V'(\phi) \equiv \partial V(\phi)/\partial \phi$ , etc. For the scalar model, we find that the spectral index  $n_s$  of the curvature perturbations, the tensor-to-scalar ratio  $r$  of the density perturbations, and the running of the spectral index  $\alpha_s$  can be expressed as

$$n_s - 1 \sim -6\epsilon + 2\eta, \quad r = 16\epsilon, \quad \alpha_s \equiv \frac{dn_s}{d \ln k} \sim 16\epsilon\eta - 24\epsilon^2 - 2\xi^2. \quad (\text{II.3})$$

We suppose the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1,2,3} (dx^i)^2. \quad (\text{II.4})$$

Here,  $a(t)$  is the scale factor and the Hubble ratio is defined as  $H \equiv \dot{a}/a$  with the dot  $\dot{\phantom{x}}$  expressing the derivative with respect to time,  $\partial/\partial t$ .

We describe the expressions of the slow-roll parameters with  $H$ . For the action in Eq. (II.1), the gravitational equations in the FLRW background in Eq. (II.4) are given by

$$\frac{3}{\kappa^2} H^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad (\text{II.5})$$

$$-\frac{1}{\kappa^2} \left( 3H^2 + 2\dot{H} \right) = \frac{1}{2}\dot{\phi}^2 - V(\phi). \quad (\text{II.6})$$

We now redefine the scalar field  $\phi$  by a new scalar field  $\varphi$ ,  $\phi = \phi(\varphi)$  and identify  $\varphi$  with the number of  $e$ -folds  $N$ . Then, Eqs. (II.5) and (II.6) can be rewritten as follows,

$$\frac{3}{\kappa^2} (H(N))^2 = \frac{1}{2} \omega(\varphi) (H(N))^2 + V(\phi(\varphi)), \quad (\text{II.7})$$

$$-\frac{1}{\kappa^2} \left( 3(H(N))^2 + 2H'(N)H(N) \right) = \frac{1}{2} \omega(\varphi) (H(N))^2 - V(\phi(\varphi)). \quad (\text{II.8})$$

Here,  $\omega(\varphi) \equiv (d\phi/d\varphi)^2$ . Therefore, if the Hubble expansion rate  $H$  is given by a function of  $N$  as  $H(N)$  and  $\omega(\varphi)$  and  $V(\varphi) \equiv V(\phi(\varphi))$  are written in the following form,

$$\omega(\varphi) = - \left. \frac{2H'(N)}{\kappa^2 H(N)} \right|_{N=\varphi}, \quad V(\varphi) = \left. \frac{1}{\kappa^2} \left[ 3(H(N))^2 + H(N)H'(N) \right] \right|_{N=\varphi}, \quad (\text{II.9})$$

we find  $H = H(N)$ ,  $\varphi = N$  as a solution for the field equation  $\phi$  or  $\varphi$  and the Einstein equation. It should be noted that  $H' < 0$  because  $\omega(\varphi) > 0$ . Thus, we can express the slow-roll parameters  $\epsilon$ ,  $\eta$  and  $\xi$  in terms of  $H$ . The representations are described in (2.8) of Ref. [10].

### B. Perfect fluid description

We now rewrite the expressions of the slow-roll parameters in terms of the perfect fluid. In the FLRW space-time (II.4), the gravitational field equations for the perfect fluid has the following form

$$\frac{3}{\kappa^2} (H(N))^2 = \rho, \quad -\frac{2}{\kappa^2} H(N)H'(N) = \rho + P, \quad (\text{II.10})$$

where  $\rho$  and  $P$  are the energy density and pressure of the perfect fluid, respectively. We assume a rather general equation of state (EoS)

$$P(N) = -\rho(N) + f(\rho), \quad (\text{II.11})$$

with  $f(\rho)$  an arbitrary function of  $\rho$ . In this case, the second equation in (II.10) has the following form:

$$-\frac{2}{\kappa^2} H(N)H'(N) = f(\rho), \quad (\text{II.12})$$

and the conservation law  $0 = \rho'(N) + 3(\rho(N) + P(N))$  reads

$$0 = \rho'(N) + 3f(\rho). \quad (\text{II.13})$$

By using Eqs. (II.12) and (II.13), we obtain

$$\frac{2}{\kappa^2} \left[ (H'(N))^2 + H(N)H''(N) \right] = 3f'(\rho)f(\rho). \quad (\text{II.14})$$

Here, it is emphasized that the prime operating  $f(\rho)$  means the derivative with respect to  $\rho$  of  $f'(\rho) \equiv df(\rho)/d\rho$ , whereas that  $H'(N) \equiv dH(N)/dN$  and  $\rho'(N) \equiv d\rho(N)/dN$ . All the slow-roll parameters can be rewritten only in terms of  $\rho(N)$  and  $f(\rho(N))$ . Hence, we obtain the expressions of observables for inflationary models,  $n_s$ ,  $r$ , and  $\alpha_s$ , shown in Appendix A.

### C. Reconstruction of perfect fluid models

As examples of the form for the square of the Hubble parameter, we examine two models. One is the linear form and the other is the exponential one.

### 1. Linear form

As the first example, we examine the following linear form for  $H^2$ :

$$(H(N))^2 = G_0 N + G_1, \quad (\text{II.15})$$

with  $G_0 (< 0)$  and  $G_1 (> 0)$  constants.

The motivations why we take such a linear form is as follows. For the slow-roll exponential inflation, the scale factor is given by  $a = \bar{a} \exp(H_{\text{inf}} t)$  with  $\bar{a}$  a constant. Here,  $H_{\text{inf}}$  is the Hubble parameter at the inflationary stage is approximately equal to a constant, namely, its time dependence is very weak. To express the weak time dependence of  $H$ , we use the form in Eq. (II.15), in which the number of  $e$ -folds is considered to play a role of time. In this case, if  $G_1/G_0 \ll N$ , the time dependence of the Hubble parameter during inflation is negligible.

Using the gravitational equations in (II.10) with Eq. (II.15) and  $H > 0$ , we obtain

$$\rho(N) = \frac{3}{\kappa^2} (G_0 N + G_1), \quad P(N) = -\frac{1}{\kappa^2} [(3N + 1) G_0 + 3G_1]. \quad (\text{II.16})$$

By eliminating  $N$  from these equations, we acquire

$$P(N) = -\rho(N) - \frac{G_0}{\kappa^2}. \quad (\text{II.17})$$

It follows from Eqs. (II.11) and (II.17) that

$$f(\rho) = -\frac{G_0}{\kappa^2}. \quad (\text{II.18})$$

Therefore, with Eqs. (II.11) and the equations in (II.16), we see that the EoS reads

$$w(N) \equiv \frac{P(N)}{\rho(N)} = -1 + \frac{f(\rho)}{\rho(N)} = -\frac{(3N + 1) G_0 + 3G_1}{3(G_0 N + G_1)}. \quad (\text{II.19})$$

### 2. Exponential form

As the second example, we study the following exponential form for  $H^2$ :

$$(H(N))^2 = G_2 e^{\beta N} + G_3, \quad (\text{II.20})$$

with  $G_2 (< 0)$ ,  $G_3 (> 0)$ , and  $\beta (> 0)$  constants.

The physical reason why we examine this exponential form is the following. In the power-law inflation, the scale factor is expressed as  $a = \bar{a} t^{\hat{p}}$  with  $\hat{p}$  a constant. In this case, the square of the the Hubble parameter during inflation becomes  $H^2 = (\dot{a}/a)^2 = \hat{p}^2 \exp(-2N/\hat{p})$ . This is equivalent to the form in Eq. (II.20) with  $G_2 = \hat{p}^2$ ,  $\beta = -2/\hat{p}$ , and  $G_3 = 0$ . Thus, such an exponential form can describe the power-law inflation.

From the gravitational equations in (II.10) with Eq. (II.20) and  $H > 0$ , we have

$$\rho(N) = \frac{3}{\kappa^2} (G_2 e^{\beta N} + G_3), \quad P(N) = -\frac{1}{\kappa^2} [(3 + \beta) G_2 e^{\beta N} + 3G_3]. \quad (\text{II.21})$$

The elimination of  $N$  from these equations gives

$$P(N) = -\left(1 + \frac{\beta}{3}\right) \rho(N) + \frac{G_3 \beta}{\kappa^2}. \quad (\text{II.22})$$

By using the first equation in (II.21) and comparing Eq. (II.22) with Eq. (II.11), we get

$$f(\rho) = -\frac{\beta}{3} \rho(N) + \frac{G_3 \beta}{\kappa^2}. \quad (\text{II.23})$$

Accordingly, from the first equality in (II.19) we find

$$w(N) = -\frac{(3 + \beta) G_2 e^{\beta N} + 3G_3}{3(G_2 e^{\beta N} + G_3)}. \quad (\text{II.24})$$

### 3. Another form

We here mention that various models of  $H(N)$  or the representations by the equation of state have been investigated in Refs. [26–28]. As an example, we show the following EoS parameter [26],

$$w = -1 + \frac{\tilde{\beta}}{(1+N)^{\tilde{\gamma}}}. \quad (\text{II.25})$$

Here,  $\{\tilde{\beta}, \tilde{\gamma}\}$  are free parameters. Then, by solving the gravitational field equations in the FLRW background, the Hubble parameter becomes

$$H = \bar{H} \exp \left[ -\frac{3\tilde{\beta}(1+N)^{1-\tilde{\gamma}}}{2(1-\tilde{\gamma})} \right], \quad (\text{II.26})$$

with  $\bar{H}$  a constant, while the spectral index and the tensor-to-scalar ratio are given by

$$n_s - 1 = \frac{1}{6} \left\{ -18\tilde{\beta}(N+1)^{-\tilde{\gamma}} - \frac{4\tilde{\gamma}^2(N+1)^{\tilde{\gamma}-2} [\tilde{\beta} + (N+1)^{\tilde{\gamma}}]}{[\tilde{\beta} - 2(N+1)^{\tilde{\gamma}}]^2} + \frac{6\tilde{\gamma}\tilde{\beta}(N+1) + 4\tilde{\gamma}(3N+2)(N+1)^{\tilde{\gamma}}}{(N+1)^2 [2(N+1)^{\tilde{\gamma}} - \tilde{\beta}]} \right\}, \quad (\text{II.27})$$

$$r = \frac{8\tilde{\beta}(N+1)^{-\tilde{\gamma}-2} [(\tilde{\gamma} - 6N - 6)(N+1)^{\tilde{\gamma}} + 3\tilde{\beta}(N+1)]^2}{3[\tilde{\beta} - 2(N+1)^{\tilde{\gamma}}]^2}. \quad (\text{II.28})$$

Hence, for the appropriate values of the free parameters  $\{\tilde{\beta}, \tilde{\gamma}\}$ , the observational values such as  $n_s$  and  $r$  can be reproduced. We note that the explicit expression of  $n_s$  is also presented in Ref. [26].

### 4. Comparison with the observations

Provided that the time variation of  $f(\rho)$  and  $\rho$  during inflation is sufficiently small, and that inflation is almost exponential as  $w(N) \equiv P(N)/\rho(N) = -1 + f(\rho)/\rho(N) \approx -1$ , i.e.,  $|f(\rho)/\rho(N)| \ll 1$ , from the expressions in Appendix A we acquire

$$n_s \sim 1 - 6 \frac{f(\rho)}{\rho(N)}, \quad r \approx 24 \frac{f(\rho)}{\rho(N)}, \quad \alpha_s \approx -9 \left( \frac{f(\rho)}{\rho(N)} \right)^2. \quad (\text{II.29})$$

We here present the recent observations on the spectral index  $n_s$ , the tensor-to-scalar ratio  $r$ , and the running of the spectral index  $\alpha_s$ . The Planck data [4, 5] suggest  $n_s = 0.9603 \pm 0.0073$  (68% CL),  $r < 0.11$  (95% CL), and  $\alpha_s = -0.0134 \pm 0.0090$  (68% CL) [the Planck and WMAP [2, 3]], the negative sign of which is at  $1.5\sigma$ . The BICEP2 experiment [1] implies  $r = 0.20^{+0.07}_{-0.05}$  (68% CL). It is mentioned that the discussions exist on how to subtract the foreground, for example, in Refs. [29–31]. Recently, there have also appeared the progresses to ensure the BICEP2 statements in Ref. [32]. It is also remarked that the representation of  $\alpha_s$  is also given in Ref. [33].

It is seen from the equations in (II.29) that when the condition  $f(\rho)/\rho(N) = 6.65 \times 10^{-3}$  is realized in the inflationary era, we find  $(n_s, r, \alpha_s) = (0.960, 0.160, -3.98 \times 10^{-4})$ . In the linear form of  $H^2$  in Eq. (II.15), if  $G_1/G_0 \gg N$  and  $-G_0/(3G_1) = 6.65 \times 10^{-3}$ , the change of value of  $w(N)$  is considered to be negligible, and the above condition can be met at the inflationary stage. On the other hand, for the exponential form of  $H^2$  in Eq. (II.20), provided that  $\beta = 2.0 \times 10^{-4}$  and hence  $\beta N \ll 1$ , and that  $-1/3 \{[1 + G_3/(G_2\beta)]\} = 6.65 \times 10^{-3}$ , then  $w(N)$  can be regarded as constant and the above condition can be satisfied during inflation. As a consequence, it is interpreted that perfect fluid models can lead to the Planck result with  $r = \mathcal{O}(0.1)$ , the value of which is compatible with the BICEP2 experiment.

Moreover, we mention concrete models [34]. For a model of  $P(N) = -\rho(N) + f(\rho)$  with  $f(\rho) = \bar{f} \sin(\rho(N)/\bar{\rho})$ , where  $\bar{f}$  is a constant and  $\bar{\rho}$  is a fiducial value of  $\rho$ , known to produce the Pseudo-Rip scenario [35], if  $\rho(N)/\bar{\rho} \ll 1$  and  $f(\rho)/\rho(N) \approx \bar{f}/\bar{\rho} = 6.65 \times 10^{-3}$ ,  $f(\rho)/\rho(N)$  behaves almost constant and the above condition can be met. In addition, we examine the other model of  $P(N) = -\rho(N) + f(\rho)$ , where  $f(\rho) = (\rho(N))^\tau$  with  $\tau (\neq 1)$  a constant. By using the first equation in (II.10), we obtain  $f(\rho)/\rho(N) \approx (3H_{\text{inf}}^2/\kappa^2)^{\tau-1}$ , where  $H_{\text{inf}} = \text{constant}$  is the Hubble parameter at the slow-roll inflation regime. Thus, similarly to the first example, when  $f(\rho)/\rho(N) \approx (3H_{\text{inf}}^2/\kappa^2)^{\tau-1} = 6.65 \times 10^{-3}$ ,  $f(\rho)/\rho(N)$  can be considered to be constant and the above condition can be satisfied.

### III. $F(R)$ GRAVITY DESCRIPTION OF THE SLOW-ROLL PARAMETERS

Next, we describe the slow-roll parameters in terms of  $F(R)$  gravity.

#### A. $F(R)$ gravity

The action of  $F(R)$  gravity is written as

$$S_{F(R)} = \int d^4x \sqrt{-g} \left( \frac{F(R)}{2\kappa^2} + \mathcal{L}_{\text{matter}} \right), \quad (\text{III.1})$$

with  $g$  the determinant of the metric tensor  $g_{\mu\nu}$ . The equation of motion for modified gravity is given by

$$\frac{1}{2}g_{\mu\nu}F(R) - R_{\mu\nu}F'(R) - g_{\mu\nu}\square F'(R) + \nabla_\mu \nabla_\nu F'(R) = -\frac{\kappa^2}{2}T_{\text{matter}\mu\nu}. \quad (\text{III.2})$$

Here,  $\nabla_\mu$  is the covariant derivative operator and  $\square \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$  is the covariant d'Alembertian. Moreover,  $T_{\text{matter}\mu\nu}$  is the energy-momentum tensor of matter. By assuming spatially flat FLRW universe, the FLRW-like equations have the following forms:

$$0 = -\frac{F(R)}{2} + 3\left(H^2 + \dot{H}\right)F'(R) - 18\left(4H^2\dot{H} + H\ddot{H}\right)F''(R) + \kappa^2\rho_{\text{matter}}, \quad (\text{III.3})$$

$$0 = \frac{F(R)}{2} - \left(\dot{H} + 3H^2\right)F'(R) + 6\left(8H^2\dot{H} + 4\dot{H}^2 + 6H\ddot{H} + \ddot{H}\right)F''(R) + 36\left(4H\dot{H} + \ddot{H}\right)^2F'''(R) + \kappa^2P_{\text{matter}}, \quad (\text{III.4})$$

where  $\rho_{\text{matter}}$  and  $P_{\text{matter}}$  are the energy density and pressure of matter, respectively, and the scalar curvature  $R$  is given by  $R = 12H^2 + 6\dot{H}$ . We note that  $F'(R) \equiv dF(R)/dR$ , while  $R'(N) \equiv dR/dN$ . From Eq. (III.3), we have

$$H^2 = \frac{-F(R) + RF'(R)}{6(F'(R) + R'(N)F''(R))}. \quad (\text{III.5})$$

In the following, to make mathematical expressions simpler, we omit the arguments of  $H(N)$ ,  $R(N)$ , and  $F(R)$ . We again remark that the prime operating  $H$  and  $R$  means the derivative with respect to  $N$ , while the one operating  $F$  denotes that with respect to  $R$ . Therefore, all of the slow-roll parameters can be expressed in terms of  $R$ ,  $R'$ ,  $R''$ ,  $R'''$ , and  $F(R)$ . Eventually, we acquire the representations of observables for inflationary models,  $n_s$ ,  $r$ , and  $\alpha_s$ , presented in Appendix B.

#### B. Reconstruction of $F(R)$ gravity models

The procedure of reconstruction of  $F(R)$  gravity has been proposed in Ref. [37]. We define the number of  $e$ -folds as  $N \equiv \ln(a/a_0)$  with  $a_0$  the scale factor at the present time  $t_0$ , and the redshift becomes  $z \equiv a_0/a - 1$ . Accordingly, we have  $N = -\ln(1+z)$ . We write the Hubble parameter in terms of  $N$  via the function  $G(N)$  as

$$(H(N))^2 = G(N) = G(-\ln(1+z)). \quad (\text{III.6})$$

Moreover, the scalar curvature is given by  $R = 3G'(N) + 12G(N)$ . For Eq. (III.6), Eq. (III.3) is rewritten to

$$0 = -9G(N(R))(4G'(N(R)) + G''(N(R)))\frac{d^2F(R)}{dR^2} + \left(3G(N(R)) + \frac{3}{2}G'(N(R))\right)\frac{dF(R)}{dR} - \frac{F(R)}{2} + \sum_i \rho_{\text{matter } i0} a_0^{-3(1+w_i)} \exp[-3(1+w_i)N(R)], \quad (\text{III.7})$$

where the last term in the right-hand side is equal to the total matter density  $\rho_{\text{matter}}$ . Here, the matters are supposed to be fluids labeled by the subscription “ $i$ ” with their constant equations of state  $w_i \equiv P_{\text{matter } i}/\rho_{\text{matter } i}$  with  $\rho_{\text{matter } i}$  and  $P_{\text{matter } i}$  the energy density and pressure of the  $i$ -th fluid, respectively, and  $\rho_{\text{matter } i0}$  is a constant. In deriving Eq. (III.7), we have used the continuity equation  $\dot{\rho}_i + 3H(1+w_i)\rho_i = 0$ . Hence, by assuming a specific form of the

Hubble parameter, i.e.,  $G(N)$ , and using the relation between the Ricci scalar and the number of  $e$ -folds, Eq. (III.7) may be solved, so that the gravitational action  $F(R)$  reproducing the expansion history described by  $H$  can be found. On the other hand, for a particular  $F(R)$  gravity model, the corresponding Hubble parameter  $G(N)$  can be obtained. Thus, with the expressions for the observables of inflationary models derived in Sec. III A, we may acquire the corresponding predictions on inflation for a particular action of  $F(R)$  gravity.

### 1. Linear form

Let us explore the linear form for  $H^2$  in Eq. (II.15). The number of  $e$ -folds can be represented in terms of the Ricci scalar as  $N(R) = (-3G_0 - 12G_1 + R) / (12G_0)$ . The Friedmann equation in vacuum becomes a second order differential equation for  $F(R)$  with respect to the Ricci scalar and its solution is derived as

$$F(R) = C_1(6G_0 - 2R)^{3/2} \sqrt{\frac{R}{12G_0} - \frac{1}{4}} \left[ 1 - \frac{1}{\frac{R}{12G_0} - \frac{1}{4}} - \frac{1}{4 \left( \frac{R}{12G_0} - \frac{1}{4} \right)^2} \right] + C_2(6G_0 - 2R)^{3/2} L \left( \frac{1}{2}, \frac{3}{2}; \frac{R}{12G_0} - \frac{1}{4} \right). \quad (\text{III.8})$$

Here,  $C_1$  and  $C_2$  are integration constants, while  $L(u_1, u_2; y)$  is the generalized Laguerre polynomial, where  $u_1$  and  $u_2$  are constants and  $y$  is a variable. As a consequence, the corresponding gravitational Lagrangian has been reconstructed. The representations of the observables for inflationary models are written in Appendix C.

The inflationary phase has to last long enough to account for the initial conditions problems, e.g., the so-called horizon and flatness problems. The value of number of  $e$ -folds at the end of inflation should be  $N_e \gtrsim 50$ . The slow-roll parameters should be much smaller than unity during inflation, whereas the slow-roll parameters become larger than or equal to unity at the end of inflation,  $N = N_e$ . As examples, if  $(N, G_0, G_1) = (50.0, -0.850, 95.0)$  and  $(60.0, -0.950, 115)$ , we obtain  $(n_s, r, \alpha_s) = (0.967, 0.121, -5.42 \times 10^{-5})$  and  $(0.967, 0.123, -5.55 \times 10^{-5})$ , respectively. Thus, the Planck results for  $n_s$  with  $r = \mathcal{O}(0.1)$  can be realized. For illustrative purposes, we may fit the free parameters  $\{G_0, G_1\}$  with the values of  $(n_s, r, \alpha_s)$  suggested by the Planck satellite and BICEP2 experiment by using the technique of the maximum likelihood, which is determined by the probability distribution:

$$P(G_i) = \mathcal{N} \exp \left( -\frac{1}{2} \chi^2(G_i) \right), \quad \text{where} \quad \chi^2(G_i) \equiv \sum_j \frac{(y_j^{\text{obs}} - y_j^{\text{th}}(G_i))^2}{\sigma_{y^{\text{obs}}}^2}, \quad (\text{III.9})$$

with  $\mathcal{N}$  a normalization factor and  $G_i$  the free parameters of the model. Here,  $\{y_j^{\text{obs}}, y_j^{\text{th}}(G_i)\}$  are the observational data and the theoretical values predicted by a particular model respectively, while  $\sigma_{y^{\text{obs}}}$  are the errors. Supposing the number of  $e$ -folds  $N_e = 50$ , the best fit is given by the minimum of the function  $\chi^2(G_i)$ . For the model (III.8), this gives  $\chi_{\text{min}}^2 = 2.322$ , which corresponds to the set of free parameters  $(G_0, G_1) = (3.03, -297.98)$ , and lead to the following inflationary parameters:

$$n_s = 0.958, \quad r = 0.16, \quad \alpha_s = -0.00088. \quad (\text{III.10})$$

The corresponding contour plot is depicted in Fig. 1, which presents large errors and correlations in both parameters as expected, because the fit is provided by using just one more observational data than the number of free parameters. The contour displays  $1\sigma$ ,  $2\sigma$ ,  $2.58\sigma$ , and  $4\sigma$  confidence levels from darker blue to lighter one.

We mention that the maximum likelihood leads to a set of free parameters  $\{G_0, G_1\}$  that give approximately the correct results for the observables of inflationary models  $(n_s, r, \alpha_s)$ . As a result, the inflationary era can be realized by the  $F(R)$  gravity model with the form in Eq. (III.8).

### 2. Exponential form

Next, we investigate the exponential form for  $H^2$  in Eq. (II.20). In this case, the relation between  $N$  and  $R$  is written as  $e^{\beta N} = (R - 12G_3) / [3G_2(4 + \beta)]$ . The solution for the Friedmann equation in vacuum becomes

$$F(R) = C_1 F(b_+, b_-, l; y) + C_2 (12G_3 - R)^{(1+1/\beta)} F \left( 1 + b_- + \frac{1}{\alpha}, 1 + b_+ + \frac{1}{\alpha}, 2 - d; y \right), \quad (\text{III.11})$$

with

$$b_{\pm} = \frac{-3\beta - 2 \pm \sqrt{\beta^2 - 20\beta + 4}}{4\beta}, \quad d = -\frac{1}{\beta}, \quad y = \frac{12G_3 - R}{12G_3 + 3G_3\beta}, \quad (\text{III.12})$$



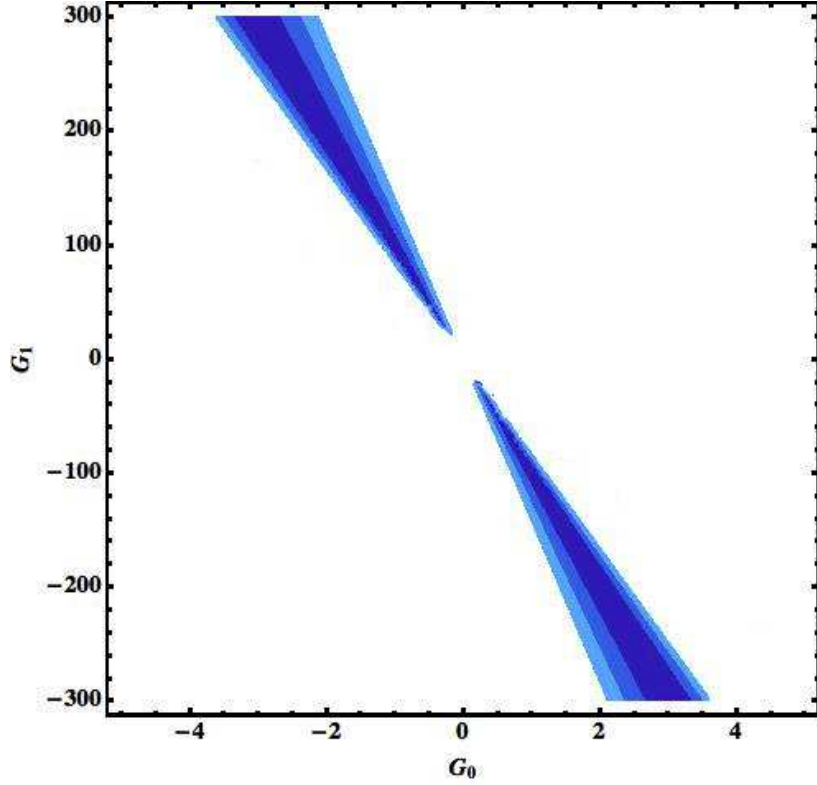


FIG. 1: Contour plot of the parameters  $G_0$  and  $G_1$  for the  $F(R)$  gravity model in Eq. (III.8). From darker blue to lighter one, the contour shows 1  $\sigma$ , 2  $\sigma$ , 2.58  $\sigma$ , and 4  $\sigma$  confidence levels.

where  $F(v_1, v_2, v_3; y)$  with  $v_j$  ( $j = 1, \dots, 3$ ) constants is the hypergeometric function.

It follows from these expressions that for  $(N, G_2, G_3) = (50.0, -1.10, 10.0)$  and  $(60.0, -1.20, 15.0)$ , we acquire  $(n_s, r, \alpha_s) = (0.963, 6.89 \times 10^{-2}, -5.06 \times 10^{-5})$  and  $(0.965, 5.84 \times 10^{-2}, -4.51 \times 10^{-5})$ , respectively. Hence, these results are compatible with the Planck data on  $n_s$  and  $r$ . In addition, by fitting the free parameters with the Planck and BICEP2 data as done in the previous model, the maximum likelihood is given by  $\chi^2_{\min} = 2.72$ , which corresponds to the free parameters  $(\beta, G_2/G_3) = (-0.013, -4.04)$ . Note that we have rewritten the Hubble parameter as  $G(N) = G_2 [e^{\beta N} + (G_3/G_2)]$ , so that we can constrain  $G_3/G_2$  in order to keep a less number of free parameters than number of data. Consequently, the best fit suggests the following values of the observables of inflationary models:

$$n_s = 0.961, \quad r = 0.20, \quad \alpha_s = 0.0013. \quad (\text{III.13})$$

As in the previous model, the contour plot at the Fig. 2 shows large errors, particularly for  $G_3/G_2$ , whereas  $\beta$  is much well constrained. However, the best fit presents quite good values for  $n_s$  and  $r$  in spite of the deviation in  $\alpha_s$ .

### C. Power-law model of $F(R)$ gravity

Now, we take a concrete  $F(R)$  gravity model and derive the observables of inflationary models. We study the power-law  $F(R)$  gravity model of the action in Eq. (III.1) with  $F(R) = \gamma R^n$ , where  $\gamma$  and  $n$  are constants. In this model, the Friedmann equation (III.7) for pure  $F(R)$  gravity is a non-linear differential equation for  $G(N)$ . This has several independent solutions, one of which is given by

$$G(N) = H^2 = G_4 e^{-2N/\zeta}, \quad \text{where} \quad \zeta = \frac{-1 + 3n - 2n^2}{n - 2}, \quad (\text{III.14})$$

with  $G_4(> 0)$  a positive constant. This form is equivalent to that in Eq. (II.20) with  $G_2 = G_4$ ,  $G_3 = 0$ , and  $\beta = -2/\zeta$ . This solution is valid for  $n \neq 2$ . By using Eq. (III.14), we find that the slow-roll parameters are represented as  $\epsilon = 1/\zeta$ ,  $\eta = 2/\zeta$ , and  $\xi^2 = -4/\zeta$ . Since the slow-roll parameters are constants, if  $\zeta \gg 1$ , we have  $\epsilon \ll 1$ ,  $\eta \ll 1$ , and



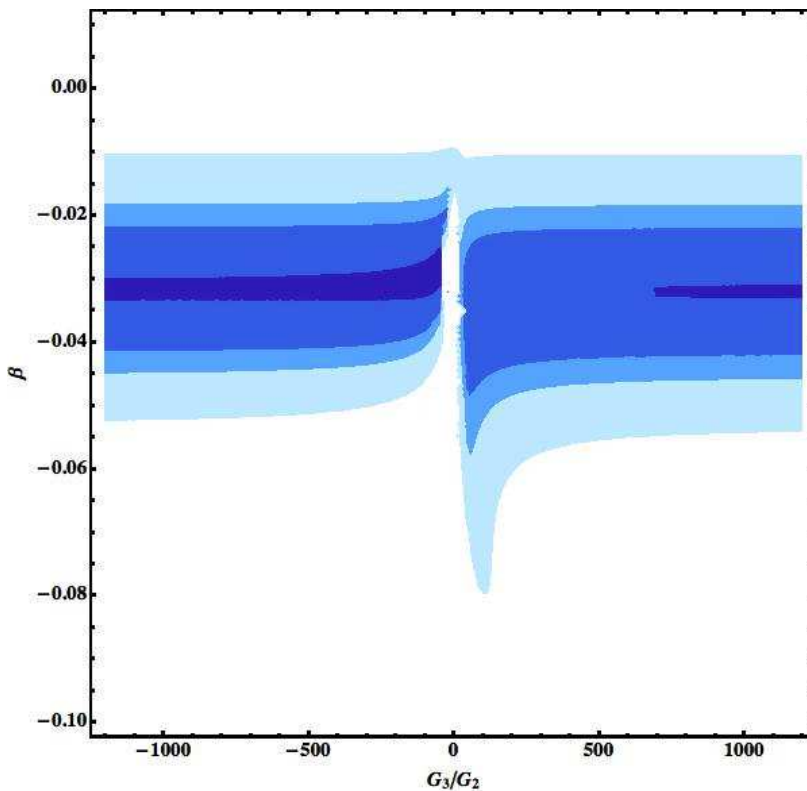


FIG. 2: Contour plot of the parameters  $\beta$  and  $G_3/G_2$  for the  $F(R)$  gravity model in Eq. (III.11). Legend is the same as Fig. 1.

$\xi^2 \ll 1$  during inflation. There is an additional solution that may lead to the slow-roll regime and the following end of inflation. The Hubble parameter for the solution reads

$$G(N) = G_4 \exp \left[ \frac{(-2+n)N + n(n-1) \ln \left( 1 - \exp \left\{ -\frac{(-5+4n)[N-2G_4n(n-1)]}{n-1} - 3n + 2n^2 \right\} \right)}{(n-1)(2n-1)} \right]. \quad (\text{III.15})$$

By a particular choice of the free parameters, one may be able to reproduce the correct observational values of  $n_s$  and/or  $r$ . By fitting the Hubble parameter (III.15) with the data, we get  $\chi^2_{\min} = 2.25$  that corresponds to  $(n, G_4) = (1.96, < 8.84)$ . Note that  $G_4$  plays an important role, while  $G_3$  in Eq. (III.15) is irrelevant for the values of the inflationary parameters. The inflationary parameters at the best fit read

$$n_s = 0.976, \quad r = 0.18, \quad \alpha_s = -4.33 \times 10^{-19}. \quad (\text{III.16})$$

As shown in Fig. 3, the value of  $n$  is well constrained, while the initial condition  $G_4$  allows a large range of the values.

We remark that the power-law model of  $F(R) = \gamma R^n$  can be regarded as a particular limit of a viable  $F(R)$  gravity model, which recovers General Relativity on small scales. For instance, in Ref. [38] there has proposed the following model:  $F(R) = R + [R^m (q_1 R^m - q_2)] / (1 + q_3 R^m)$  with  $m$  and  $q_j$  ( $j = 1, \dots, 3$ ) constants. At the inflationary stage, where the curvature is expected to be very large, this model tends to the power-law one, so that the above analysis can be applied for this particular viable  $F(R)$  gravity model.

#### IV. CONCLUSIONS

In the present paper, we have reconstructed the descriptions of inflation in perfect fluid models and  $F(R)$  gravity theories. Particularly, as the observables of inflationary models, we have explored the spectral index of curvature perturbations, the tensor-to-scalar ratio, and the running of the spectral index. In addition, we have compared the perfect fluid and  $F(R)$  gravity descriptions with the Planck and BICEP2 results.

In the perfect fluid description, if the equation of state for the perfect fluid is approximately equal to minus unity, namely, the difference of the value between the equation of state and the cosmological constant is much smaller than

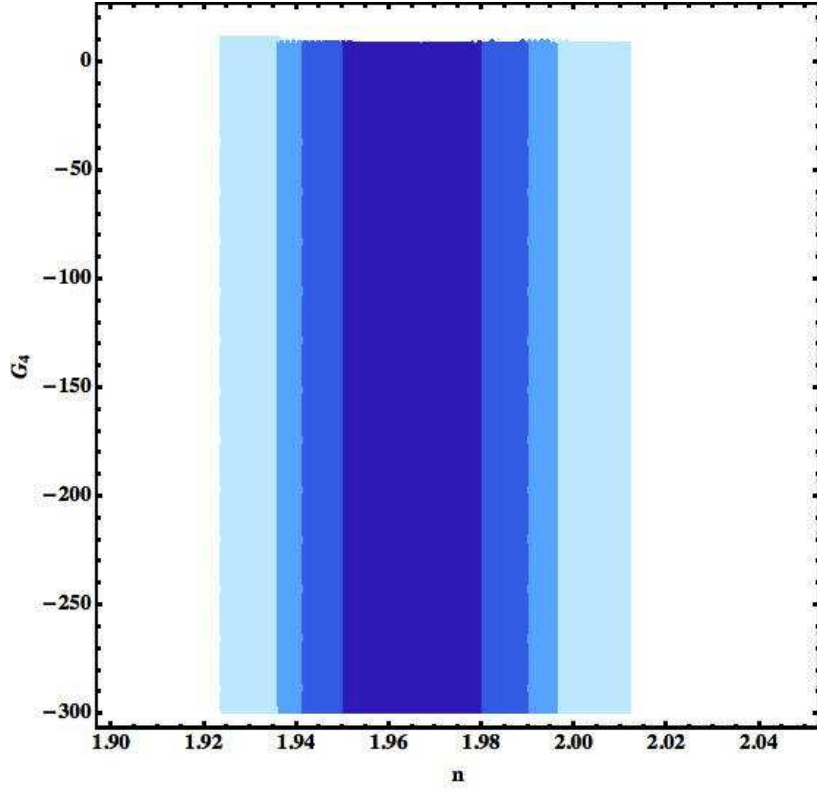


FIG. 3: Contour plot of the parameters  $n$  and  $G_4$  for the  $F(R)$  gravity model of  $F(R) = \gamma R^n$ . Legend is the same as Fig. 1.

unity, the Planck result for the spectral index of curvature perturbations with the tensor-to-scalar ratio of order of 0.1 can be reproduced.

On the other hand, in the description of  $F(R)$  gravity, if the squared of the Hubble parameter has a linear form in terms of the number of  $e$ -folds during inflation, the spectral index is consistent with the Planck data and the tensor-to-scalar ratio becomes  $\mathcal{O}(0.1)$ , where as the Hubble parameter is given by an exponential function in terms of the number of  $e$ -folds, the value of the spectral index and the tensor-to-scalar ratio is compatible with the Planck analysis. By fitting the above  $F(R)$  gravity models with the observational data released by Planck satellite and BICEP2 experiment, we have found that the best fit corresponds to the third model,  $F(R) \propto R^n$ , which provides the maximum likelihood according to the value of  $\chi^2_{\min}$ . Regarding this model, the predicted values for  $n_s$  and  $r$  are within the allowed ranges suggested by the Planck and BICEP2 data. In spite of the great uncertainty of the integration constant  $G_4$  and therefore on the initial conditions for the Hubble parameter, it provides a good constraint on the value of  $n = 1.96^{+0.02}_{-0.01}$ .

It is finally remarked that the reconstruction method developed in this work may easily be extended for the case that there exists convenient matter. The corresponding analysis should not lead to any significant qualitative change in comparison with the analysis for pure gravity. It will be executed elsewhere.

### Acknowledgments

We would like to thank Professor Jaume de Haro for valuable discussions and suggestions very much. This work was partially supported by MINECO (Spain) project FIS2010-15640 (S.D.O. and D.S.-G.), the JSPS Grant-in-Aid for Scientific Research (S) # 22224003 and (C) # 23540296 (S.N.), that for Young Scientists (B) # 25800136 (K.B.), the University of the Basque Country, Project Consolider CPAN Bo. CSD2007-00042 and the NRF financial support from the University of Cape Town (South Africa) (D.S.-G.).

### Appendix A: Expressions of observables of inflationary models for the description of the perfect fluid

In this Appendix, we present the observables of inflationary models in the description of the perfect fluid. Those expressions become

$$\begin{aligned}
n_s &\sim 1 - 9\rho(N)f(\rho) \left( \frac{f'(\rho) - 2}{2\rho(N) - f(\rho)} \right)^2 + \frac{6\rho(N)}{2\rho(N) - f(\rho)} \left\{ \frac{f(\rho)}{\rho(N)} + \frac{1}{2} (f'(\rho))^2 + f'(\rho) - \frac{5}{2} \frac{f(\rho)f'(\rho)}{\rho(N)} + \left( \frac{f(\rho)}{\rho(N)} \right)^2 \right. \\
&\quad \left. + \frac{1}{3} \frac{\rho'(N)}{f(\rho)} \left[ (f'(\rho))^2 + f(\rho)f''(\rho) - 2 \frac{f(\rho)f'(\rho)}{\rho(N)} + \left( \frac{f(\rho)}{\rho(N)} \right)^2 \right] \right\}, \\
r &= 24\rho(N)f(\rho) \left( \frac{f'(\rho) - 2}{2\rho(N) - f(\rho)} \right)^2, \\
\alpha_s &= \rho(N)f(\rho) \left( \frac{f'(\rho) - 2}{2\rho(N) - f(\rho)} \right)^2 \left[ \frac{72\rho(N)}{2\rho(N) - f(\rho)} J_1 - 54\rho(N)f(\rho) \left( \frac{f'(\rho) - 2}{2\rho(N) - f(\rho)} \right)^2 - \frac{1}{f'(\rho) - 2} J_2 \right],
\end{aligned}$$

where

$$\begin{aligned}
J_1 &\equiv \frac{f(\rho)}{\rho(N)} + \frac{1}{2} (f'(\rho))^2 + f'(\rho) - \frac{5}{2} \frac{f(\rho)f'(\rho)}{\rho(N)} + \left( \frac{f(\rho)}{\rho(N)} \right)^2 \\
&\quad + \frac{1}{3} \frac{\rho'(N)}{f(\rho)} \left[ (f'(\rho))^2 + f(\rho)f''(\rho) - 2 \frac{f(\rho)f'(\rho)}{\rho(N)} + \left( \frac{f(\rho)}{\rho(N)} \right)^2 \right], \\
J_2 &\equiv \frac{45}{2} \frac{f(\rho)}{\rho(N)} \left( f'(\rho) - \frac{1}{2} \frac{f(\rho)}{\rho(N)} \right) + 18 \left( \frac{f(\rho)}{\rho(N)} \right)^{-1} \left( f'(\rho) - \frac{1}{2} \frac{f(\rho)}{\rho(N)} \right)^2 + 18 \left( \frac{f(\rho)}{\rho(N)} \right)^{-1} \left( f'(\rho) - \frac{1}{2} \frac{f(\rho)}{\rho(N)} \right)^3 \\
&\quad - 9 \left( f'(\rho) - \frac{1}{2} \frac{f(\rho)}{\rho(N)} \right)^2 - 45f'(\rho) + 9 \frac{f(\rho)}{\rho(N)} \\
&\quad + 3 \left( 4f'(\rho) - 7 \frac{f(\rho)}{\rho(N)} + 2 \right) \left\{ -\frac{3}{2} \left( f'(\rho) - \frac{1}{2} \frac{f(\rho)}{\rho(N)} \right) \right. \\
&\quad \left. + \left( \frac{f(\rho)}{\rho(N)} \right)^{-2} \frac{\rho'(N)}{\rho(N)} \left[ (f'(\rho))^2 + f(\rho)f''(\rho) - 2 \frac{f(\rho)f'(\rho)}{\rho(N)} + \left( \frac{f(\rho)}{\rho(N)} \right)^2 \right] \right\} \\
&\quad + \left( \frac{f(\rho)}{\rho(N)} \right)^{-2} \left\{ -\frac{3}{2} \left( \frac{f(\rho)}{\rho(N)} \right) \left( \frac{\rho'(N)}{\rho(N)} \right) \left[ 3(f'(\rho))^2 + 2f(\rho)f''(\rho) - \frac{11}{2} \frac{f(\rho)f'(\rho)}{\rho(N)} + \frac{5}{2} \left( \frac{f(\rho)}{\rho(N)} \right)^2 \right] \right. \\
&\quad \left. + \left( \frac{\rho''(N)}{\rho(N)} \right) \left[ (f'(\rho))^2 + f(\rho)f''(\rho) - 2 \frac{f(\rho)f'(\rho)}{\rho(N)} + \left( \frac{f(\rho)}{\rho(N)} \right)^2 \right] \right. \\
&\quad \left. + \left( \frac{\rho'(N)}{\rho(N)} \right)^2 \left[ (3f'(\rho)f''(\rho) + f(\rho)f'''(\rho))\rho(N) - 3(f'(\rho))^2 - 3f(\rho)f''(\rho) \right. \right. \\
&\quad \left. \left. + 6 \frac{f(\rho)f'(\rho)}{\rho(N)} - 3 \left( \frac{f(\rho)}{\rho(N)} \right)^2 \right] \right\}.
\end{aligned}$$

### Appendix B: Expressions of observables of inflationary models for the $F(R)$ gravity description

In this Appendix, we write the observables of inflationary models in the  $F(R)$  gravity description. The representations read

$$\begin{aligned}
n_s &\sim 1 - \frac{3(F - RF')(F''R'(R' + 2R) + F'(R' - 2R) - 4F)^2}{2(-RF''R' - 2RF' + F)^2(RF''R' - RF' + 2F)} \\
&\quad - \frac{1}{2(F - RF')^2(RF''R' - RF' + 2F)^2(RF''R' + 2RF' - F)} ((RF' - F) \\
&\quad \times ((RF''R' - RF' + 2F)^4 - 18(F - RF')(RF''R' - RF' + 2F)^3
\end{aligned}$$

$$\begin{aligned}
& + \left( 11F^2 + R \left( 6R(F'')^2 (R')^2 - F'(RF' + 4F) + 6F''R'(3F - RF') \right) \right) \\
& \left( -F''(R')^2 (R^2 F'' + F) + 4F^2 + R(F')^2 (R' + 3R) - F'(R'(RF''(2R - R') + F) + 8FR) \right) \\
& + 2(RF' - F)(R'(F''R' + F'))(-2RF''R' - 3RF' + 2F) \\
& \quad (RF''R' - RF' + 2F) \left( 4F^2 + R \left( 3R(F'')^2 (R')^2 + 7R(F')^2 + F'(6RF''R' - 8F) \right) \right) \\
& - \frac{F - RF'}{F - RF'} \\
& + (RF' - F) \left( F'''(R')^3 + F''R'(2R'' + R') + F'R'' \right) (RF''R' - RF' + 2F) \Big), \\
r = & - \frac{4(-F + RF')}{2F - RF' + RR'F''} \left[ \frac{R'F' + (R')^2 F'' - 4F - 2RF' + 2RR'F''}{-F + 2RF' + RR'F''} \right]^2, \\
\alpha_s = & \frac{1}{2(F - RF')^2 (F - 2RF' - RR'F'')^4 (2F - RF' + RR'F'')^4} ((RF' - F) \\
& \times \left( 3(F - RF')^3 (2F - RF' + RR'F'')^2 (-4F + F'(R' - 2R) + R'(2R + R')F'')^4 \right. \\
& + 2(RF' - F)(2F - RF' + RR'F'')(-F + 2RF' + RR'F'') \left( (2F - RF' + RR'F'')^4 \right. \\
& - 18(F - RF')(2F - RF' + RR'F'')^3 + 2(RF' - F)(R'(F' + R'F''))(2F - 3RF' - 2RR'F'') \\
& \left. \left. (2F - RF' + RR'F'') \left( 4F^2 + R \left( 7R(F')^2 + (6RR'F'' - 8F)F' + 3R(R')^2 (F'')^2 \right) \right) \right) \right) \\
& - \frac{F - RF'}{F - RF'} \\
& + (RF' - F) \left( F'''(R')^3 + F''(R' + 2R'')R' + F'R'' \right) (2F - RF' + RR'F'') \\
& + \left( 11F^2 + R \left( 6R(R')^2 (F'')^2 + 6(3F - RF')R'F'' - F'(4F + RF') \right) \right) \left( 4F^2 + R(F')^2 (3R + R') \right. \\
& \left. - (R')^2 F''(F''R^2 + F) - F'(8FR + R'(F + R(2R - R')F'')) \right) (-4F + F'(R' - 2R) + R'(2R + R')F'')^2 \\
& + (F - RF')(F - 2RF' - RR'F'')^2 (4F + F'(R' - 2R) + R'(2R + R')F'') (256F^5 \\
& + R^2 (F')^5 \left( (R')^3 + 6R(R')^2 + 2R(3R + R'')R' + R^2(-16R + 6R'' + R''') \right) \\
& + R(F')^4 \left( R(F''''R^2 + 2F'''R + 3F'') (R')^4 + (9F'''R^3 + 8F''R^2 - 2F)(R')^3 \right. \\
& + R(R(5RR''F''' - 2F''(R - 2R'')) - 23F)(R')^2 + 2R(R^2F''(32R + 7R'') - F(17R + 4R''))R' \\
& + 3FR^2(48R - 9R'' - 2R''') + R'(-8R^4(R')^5 (F'')^5 \\
& + FR^2(R')^3 (-2(R')^2 - 39RR' + 4R(4R - 5R''))(F'')^4 \\
& + F(R')^2 \left( (F - 11R^3F''') (R')^3 - 13FR(R')^2 - FR(53R + 4R'')R' + FR^2(128R - 83R'' + 2R''') \right) (F'')^3 \\
& + F^2R' \left( R(RF'''' - 2F''') (R')^4 - 46R^2F'''(R')^3 + (5R^2R''F''' - 18F)(R')^2 - 8FR''R' \right. \\
& + 4FR(96R - 23R'' + 2R''') (F'')^2 + 4F^3 \left( (RF'''' - F''') (R')^4 - 13RF'''(R')^3 + 5RR''F'''(R')^2 \right. \\
& + 7FR' + F(128R - 3R'' + 2R''') F'' + 4F^4R' \left( F''''(R')^2 - 2F'''R' + 5R''F''' \right) \Big) \\
& - F' \left( R(F'')^2 \left( -2R^2(F'')^2 + (2F - 11R^3F''')F'' + 2FR(RF'''' - 2F''') \right) (R')^6 \right. \\
& - F'' \left( 13(F'')^3 R^4 + 21F(F'')^2 R^2 + 2F^2(4F'''' - 5RF''''')R + FF''(59F'''R^3 + 3F) \right) (R')^5 \\
& + \left( 4R^4(4R - 5R'')(F'')^4 - FR^2(19R + 8R'')(F'')^3 - 5FR(5F - 2R^2R''F''') (F'')^2 - 64F^2R^2F'''F'' \right. \\
& + 4F^3(F''' + 3RF''''') (R')^4 + F \left( R^3(192R - 97R'' + 4R''') (F'')^3 + 2FR(25R - 7R'')(F'')^2 \right. \\
& + 2F(25R''F'''R^2 + 14F)F'' + 20F^2RF'''' (R')^3 + F^2 \left( R^2(768R - 71R'' + 19R''') (F'')^2 \right. \\
& \left. + 4F(14R + 3R'')F'' + 60FRR''F''') (R')^2 + 4F^3(RF''(320R + 23R'' + 5R''') - 8F)R' \right)
\end{aligned}$$

$$\begin{aligned}
& + 4F^4 (192R + 3R'' - R''') - (F')^3 \left( R^2 F'' (2R^2 F'''' - 3F''') (R')^5 \right. \\
& + R \left( 8R^2 (F'')^2 + (7F''' R^3 + 6F) F'' + 2FR (4F''' + 3RF''''') \right) (R')^4 \\
& + (R^2 (F'' (45F + RF'' (17R + 2R'')) + R (43F + 10RF'' R'') F''') - F^2) (R')^3 \\
& + R \left( (F'')^2 (96R + 26R'' + 3R''') R^3 + FF'' (11R + 20R'') R + 3F (10R^2 R'' F''' - 9F) \right) (R')^2 \\
& + FR (R^2 F'' (448R + 97R'' + 4R''') - 2F (36R + 5R'')) R' + F^2 R^2 (512R - 33R'' - 13R''') \\
& + (F')^2 \left( R^2 (F'')^2 (F'' + R (RF'''' - 2F''')) (R')^6 - RF'' \left( 8R^2 (F'')^2 + (13F''' R^3 + 6F) F'' \right. \right. \\
& + 4FR (F''' - 2RF''''') (R')^5 + \left. \left( 4R^3 (2R - R'') (F'')^3 + R^2 (5R'' F''' R^2 + F) (F'')^2 \right. \right. \\
& + F (3F - 5R^3 F''') F'' + F^2 R (10F''' + 13RF''''') (R')^4 + R \left( 2R^3 (32R - 7R'' + R''') (F'')^3 \right. \\
& + FR (65R - 4R'') (F'')^2 + 5F (8R'' F''' R^2 + 13F) F'' + 62F^2 RF''''') (R')^3 \\
& + F \left( (F'')^2 (480R + 47R'' + 14R''') R^3 + FF'' (51R + 28R'') R + 5F (13R^2 R'' F''' - 2F) \right) (R')^2 \\
& \left. \left. + F^2 (R^2 F'' (187R'' + 16 (72R + R''')) - 4F (18R + R'')) R' + 4F^3 R (224R - 3R''') \right) \right) \right) .
\end{aligned}$$

### Appendix C: Linear and exponential forms of the Hubble parameter

In this Appendix, we write the expressions of the observables for inflationary models in the linear form of  $H^2$  in Eq. (II.15) and those in the exponential form of  $H^2$  in Eq. (II.20). For the linear form, we find

$$\begin{aligned}
n_s &= \frac{36 (G_0 N + G_1)^2 + 84 G_0 (G_0 N + G_1) - 5 G_0^2}{[6 (G_0 N + G_1) + G_0]^2}, \\
r &= -\frac{268 G_0 (G_0 N + G_1)}{[6 (G_0 N + G_1) + G_0]^2}, \\
\alpha_s &= -\frac{27 G_0^2}{2 (G_0 N + G_1)^2 [6 (G_0 N + G_1) + G_0]^4} \left\{ 64 (G_0 N + G_1)^3 [3 (G_0 N + G_1) - G_0] \right. \\
&\quad \left. - 3 G_0 [2 (G_0 N + G_1) - G_0] [6 (G_0 N + G_1) + G_0]^2 \right\}.
\end{aligned}$$

On the other hand, for the exponential form, we obtain

$$\begin{aligned}
n_s &= \frac{-6 (\beta^2 + 6\beta - 6) G_3^2 + 2 (\beta^3 + 6\beta^2 + 6\beta + 36) G_3 G_2 e^{\beta N} + (\beta^3 + 13\beta^2 + 48\beta + 36) G_2^2 e^{2\beta N}}{[6G_3 + (6 + \beta)G_2 e^{\beta N}]^2}, \\
r &= -\frac{8\beta (6 + \beta)^2 G_2 e^{\beta N} (G_3 + G_2 e^{\beta N})}{[6G_3 + (6 + \beta)G_2 e^{\beta N}]^2}, \\
\alpha_s &= \frac{\beta^2 (6 + \beta)^2 G_2 e^{\beta N}}{[6G_3 + (6 + \beta)G_2 e^{\beta N}]^4} \left\{ 4 (6 + \beta) (G_3 + G_2 e^{\beta N}) (G_3 + 2G_2 e^{\beta N}) [6G_3 + (6 + \beta)G_2 e^{\beta N}] \right. \\
&\quad - 6 (6 + \beta)^2 G_2 e^{\beta N} (G_3 + G_2 e^{\beta N})^2 - \frac{1}{2} G_2 e^{\beta N} [6G_3 + (6 + \beta)G_2 e^{\beta N}]^2 \left[ -\frac{3}{4} \beta^2 \left( \frac{G_2 e^{\beta N}}{G_3 + G_2 e^{\beta N}} \right)^2 \right. \\
&\quad + \frac{3}{2} \beta (1 + \beta - \beta^2) \left( \frac{G_2 e^{\beta N}}{G_3 + G_2 e^{\beta N}} \right) + \left( -9 + \frac{15}{2} \beta + \beta^2 + 6\beta^3 \right) \\
&\quad \left. \left. + 6\beta (1 - \beta^2) \left( \frac{G_2 e^{\beta N}}{G_3 + G_2 e^{\beta N}} \right)^{-1} \right] \right\}.
\end{aligned}$$

### Appendix D: Relation between the EoS parameter and the tensor-to-scalar ratio

In this Appendix, we explore the relation between the EoS parameter and the tensor-to-scalar ratio. In Ref. [36], Eq. (1.3) gives an expression for the tensor-to-scalar ratio:

$$r(k) = 64\pi \left( \frac{\dot{\phi}^2}{H^2} \right)_{k=aH}, \quad (\text{D1})$$

where a unit of  $G = 1$  has maybe been used. Then, Eq. (D1) can be rewritten as

$$r(k) = 8\kappa^2 \left( \frac{\dot{\phi}^2}{H^2} \right)_{k=aH}. \quad (\text{D2})$$

For the scalar field, the EoS parameter  $w$  is given by

$$w = \frac{P}{\rho}, \quad P = \frac{1}{2}\dot{\phi}^2 - V(\phi), \quad \rho = \frac{1}{2}\dot{\phi}^2 + V(\phi). \quad (\text{D3})$$

By using the Friedmann equation in the FLRW background

$$H^2 = \frac{\kappa^2}{3}\rho, \quad (\text{D4})$$

we find

$$V(\phi) = \left( \frac{24}{r(k)} - \frac{1}{2} \right) \dot{\phi}^2, \quad (\text{D5})$$

and therefore

$$H^2 = \frac{\kappa^2}{3} \frac{24}{r(k)} \dot{\phi}^2. \quad (\text{D6})$$

Thus,  $w$  can be expressed in term of  $r(k)$  as follows,

$$w = -1 + \frac{r(k)}{24}. \quad (\text{D7})$$

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- [1] P. A. R. Ade *et al.* [BICEP2 Collaboration], Phys. Rev. Lett. **112** (2014) 241101 [arXiv:1403.3985 [astro-ph.CO]].
  - [2] D. N. Spergel *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **148** (2003) 175 [arXiv:astro-ph/0302209];  
D. N. Spergel *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **170** (2007) 377 [arXiv:astro-ph/0603449];  
E. Komatsu *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **180** (2009) 330 [arXiv:0803.0547 [astro-ph]];  
Astrophys. J. Suppl. **192** (2011) 18 [arXiv:1001.4538 [astro-ph.CO]].
  - [3] G. Hinshaw *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **208** (2013) 19 [arXiv:1212.5226 [astro-ph.CO]].
  - [4] P. A. R. Ade *et al.* [Planck Collaboration], arXiv:1303.5076 [astro-ph.CO].
  - [5] P. A. R. Ade *et al.* [Planck Collaboration], arXiv:1303.5082 [astro-ph.CO].
  - [6] J. E. Lidsey, A. R. Liddle, E. W. Kolb, E. J. Copeland, T. Barreiro and M. Abney, Rev. Mod. Phys. **69** (1997) 373 [astro-ph/9508078].
  - [7] D. K. Hazra, A. Shafieloo, G. F. Smoot and A. A. Starobinsky, arXiv:1405.2012 [astro-ph.CO];  
Y. Wan, S. Li, M. Li, T. Qiu, Y. Cai and X. Zhang, arXiv:1405.2784 [astro-ph.CO].
  - [8] J. Joergensen, F. Sannino and O. Svendsen, Phys. Rev. D **90** (2014) 043509 [arXiv:1403.3289 [hep-ph]];  
T. Kobayashi and O. Seto, Phys. Rev. D **89** (2014) 103524 [arXiv:1403.5055 [astro-ph.CO]];  
A. Ashoorioon, K. Dimopoulos, M. M. Sheikh-Jabbari and G. Shiu, arXiv:1403.6099 [hep-th];  
M. S. Sloth, arXiv:1403.8051 [hep-ph];  
C. Cheng and Q. -G. Huang, arXiv:1404.1230 [astro-ph.CO];  
C. Cheng, Q. -G. Huang and W. Zhao, arXiv:1404.3467 [astro-ph.CO];  
A. Codello, J. Joergensen, F. Sannino and O. Svendsen, arXiv:1404.3558 [hep-ph];  
B. Hu, J. -W. Hu, Z. -K. Guo and R. -G. Cai, arXiv:1404.3690 [astro-ph.CO];

- T. Kobayashi, O. Seto and Y. Yamaguchi, arXiv:1404.5518 [hep-ph];  
Y. Hamada, H. Kawai and K. -y. Oda, arXiv:1404.6141 [hep-ph];  
T. Higaki, T. Kobayashi, O. Seto and Y. Yamaguchi, arXiv:1405.0775 [hep-ph];  
Q. Gao, Y. Gong and T. Li, arXiv:1405.6451 [gr-qc];  
L. Barranco, L. Boubekeur and O. Mena, arXiv:1405.7188 [astro-ph.CO];  
J. Martin, C. Ringeval, R. Trotta and V. Vennin, arXiv:1405.7272 [astro-ph.CO];  
J. Garcia-Bellido, D. Roest, M. Scalisi and I. Zavala, arXiv:1405.7399 [hep-th];  
M. Wail Hossain, R. Myrzakulov, M. Sami and E. N. Saridakis, arXiv:1405.7491 [gr-qc];  
T. Inagaki, R. Nakanishi and S. D. Odintsov, arXiv:1408.1270 [gr-qc];  
E. Elizalde, S. D. Odintsov, E. O. Pozdeeva and S. Y. Vernov, arXiv:1408.1285 [hep-th];  
M. Dine and L. Stephenson-Haskins, arXiv:1408.0046 [hep-ph];  
Y. Hamada, H. Kawai, K. Oda, S. C. Park, arXiv:1408.4864 [hep-ph].
- [9] Q. Gao and Y. Gong, Phys. Lett. B **734** (2014) 41 [arXiv:1403.5716 [gr-qc]].
- [10] K. Bamba, S. Nojiri and S. D. Odintsov, Phys. Lett. B **737** (2014) 374 [arXiv:1406.2417 [hep-th]].
- [11] S. Nojiri and S. D. Odintsov, Phys. Rept. **505** (2011) 59 [arXiv:1011.0544 [gr-qc]];  
S. Nojiri and S. D. Odintsov, eConf C **0602061** (2006) 06 [Int. J. Geom. Meth. Mod. Phys. **4** (2007) 115] [hep-th/0601213];  
K. Bamba and S. D. Odintsov, arXiv:1402.7114 [hep-th];  
S. Capozziello and V. Faraoni, *Beyond Einstein Gravity* (Springer, Dordrecht, 2010);  
S. Capozziello and M. De Laurentis, Phys. Rept. **509** (2011) 167 [arXiv:1108.6266 [gr-qc]];  
A. de la Cruz-Dombriz and D. Saez-Gomez, Entropy **14** (2012) 1717 [arXiv:1207.2663 [gr-qc]].
- [12] K. Bamba, S. Capozziello, S. Nojiri and S. D. Odintsov, Astrophys. Space Sci. **342** (2012) 155 [arXiv:1205.3421 [gr-qc]].
- [13] A. Joyce, B. Jain, J. Khoury and M. Trodden, arXiv:1407.0059 [astro-ph.CO].
- [14] R. Myrzakulov and L. Sebastiani, arXiv:1410.3573 [gr-qc].
- [15] R. H. Brandenberger, arXiv:1206.4196 [astro-ph.CO];  
R. H. Brandenberger, arXiv:1203.6698 [astro-ph.CO];  
R. H. Brandenberger, PoS ICFI **2010** (2010) 001 [arXiv:1103.2271 [astro-ph.CO]].
- [16] K. Bamba, A. N. Makarenko, A. N. Myagky, S. Nojiri and S. D. Odintsov, JCAP01(2014)008 [arXiv:1309.3748 [hep-th]].
- [17] M. Rinaldi, G. Cognola, L. Vanzo and S. Zerbini, JCAP **1408** (2014) 015 [arXiv:1406.1096 [gr-qc]].
- [18] A. A. Starobinsky, Phys. Lett. B **91**, 99 (1980).
- [19] S. Ferrara, A. Kehagias and A. Riotto, Fortsch. Phys. **62** (2014) 573 [arXiv:1403.5531 [hep-th]];  
S. Ferrara and M. Porrati, Phys. Lett. B **737** (2014) 135 [arXiv:1407.6164 [hep-th]].
- [20] L. Sebastiani, G. Cognola, R. Myrzakulov, S. D. Odintsov and S. Zerbini, Phys. Rev. D **89** (2014) 023518 [arXiv:1311.0744 [gr-qc]];  
G. K. Chakravarty and S. Mohanty, arXiv:1405.1321 [hep-ph].
- [21] K. Bamba, R. Myrzakulov, S. D. Odintsov and L. Sebastiani, Phys. Rev. D **90** (2014) 043505 [arXiv:1403.6649 [hep-th]].
- [22] K. Bamba, G. Cognola, S. D. Odintsov and S. Zerbini, Phys. Rev. D **90** (2014) 023525 [arXiv:1404.4311 [gr-qc]];  
G. Cognola, E. Elizalde and S. Zerbini, arXiv:1408.1766 [hep-th];  
M. Rinaldi, G. Cognola, L. Vanzo and S. Zerbini, arXiv:1410.0631 [gr-qc];  
J. Amorós, J. de Haro and S. D. Odintsov, Phys. Rev. D **89** (2014) 104010 [arXiv:1402.3071 [gr-qc]].
- [23] A. B. Lahanas and K. Tamvakis, arXiv:1405.0828 [hep-th];  
N. Takeda and Y. Watanabe, arXiv:1405.3830 [astro-ph.CO];  
C. Pallis, arXiv:1407.8522 [hep-ph].
- [24] D. I. Kaiser, Phys. Lett. B **340** (1994) 23 [astro-ph/9405029].
- [25] D. I. Kaiser, Phys. Rev. D **52** (1995) 4295 [astro-ph/9408044];  
D. I. Kaiser and E. I. Sfakianakis, Phys. Rev. Lett. **112** (2014) 011302 [arXiv:1304.0363 [astro-ph.CO]];  
K. Schutz, E. I. Sfakianakis and D. I. Kaiser, Phys. Rev. D **89** (2014) 064044 [arXiv:1310.8285 [astro-ph.CO]].
- [26] V. Mukhanov, Eur. Phys. J. C **73** (2013) 2486 [arXiv:1303.3925 [astro-ph.CO]].
- [27] J. Garcia-Bellido and D. Roest, arXiv:1402.2059 [astro-ph.CO].
- [28] V. Mukhanov, arXiv:1409.2335 [astro-ph.CO].
- [29] P. A. R. Ade *et al.* [Planck Collaboration], arXiv:1405.0871 [astro-ph.GA];  
P. A. R. Ade *et al.* [Planck Collaboration], arXiv:1405.0874 [astro-ph.GA];  
R. Adam *et al.* [Planck Collaboration], arXiv:1409.5738 [astro-ph.CO].
- [30] M. J. Mortonson and U. Seljak, arXiv:1405.5857 [astro-ph.CO].
- [31] M. Kamionkowski and E. D. Kovetz, arXiv:1408.4125 [astro-ph.CO].
- [32] W. N. Colley and J. R. Gott, arXiv:1409.4491 [astro-ph.CO].
- [33] B. A. Bassett, S. Tsujikawa and D. Wands, Rev. Mod. Phys. **78** (2006) 537 [astro-ph/0507632].
- [34] A. V. Astashenok, S. Nojiri, S. D. Odintsov and R. J. Scherrer, Phys. Lett. B **713** (2012) 145 [arXiv:1203.1976 [gr-qc]].
- [35] P. H. Frampton, K. J. Ludwick and R. J. Scherrer, Phys. Rev. D **84** (2011) 063003 [arXiv:1106.4996 [astro-ph.CO]];  
P. H. Frampton, K. J. Ludwick and R. J. Scherrer, Phys. Rev. D **85** (2012) 083001 [arXiv:1112.2964 [astro-ph.CO]];  
A. V. Astashenok, S. Nojiri, S. D. Odintsov and A. V. Yurov, Phys. Lett. B **709** (2012) 396 [arXiv:1201.4056 [gr-qc]].
- [36] J. A. Vazquez, M. Bridges, Y. -Z. Ma and M. P. Hobson, JCAP **1308** (2013) 001 [arXiv:1303.4014 [astro-ph.CO]].
- [37] S. Nojiri and S. D. Odintsov, Phys. Rev. D **74** (2006) 086005 [hep-th/0608008];  
S. Nojiri, S. D. Odintsov and D. Saez-Gomez, Phys. Lett. B **681** (2009) 74 [arXiv:0908.1269 [hep-th]].
- [38] S. Nojiri and S. D. Odintsov, Phys. Rev. D **77** (2008) 026007 [arXiv:0710.1738 [hep-th]].